# Habilitation à diriger des recherches

# Contributions to nonparametric hypotheses testing and statistical learning

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2 décembre 2015

# Hypotheses testing theory for concrete problems

**A theory in response to concrete challenges in various fields**

- Laser vibrometry
- Public statistics
- Genetics
- Neuroscience

# Single tests of single null hypotheses

Observed random variable: $\mathbf{X}$, defined on $(\Omega, \mathcal{A}, \mathbb{P})$, with distribution $P$.

Possible set of distributions for $\mathbf{X}$ defined from a nonparametric model: $\mathcal{P}$.

Single null hypothesis defined through $\mathcal{P}_0 \subset \mathcal{P}$ as $(H_0)\ P \in \mathcal{P}_0$.

Alternative hypothesis $(H_1)\ P \in \mathcal{P} \setminus \mathcal{P}_0$.

A (single) **nonrandomized test** of $(H_0)$ against $(H_1)$ is a statistic $\phi$ depending on $\mathbf{X}$:

- with value 1 when $\mathbf{X}$ leads to reject $(H_0)$ in favor of $(H_1)$,
- with value 0 otherwise.

# Single tests of single null hypotheses

## Nonasymptotic minimax testing

- First kind error requirement (Neyman-Pearson): given $\alpha$ in $(0,1)$, $\sup_{P\in\mathcal{P}_0} P(\phi=1) := \mathbb{P}_{(H_0)}(\phi=1) \leq \alpha$ (level $\alpha$ test).
- Second kind error requirement: given $\beta$ in $(0,1)$, $\sup_{P\in\mathcal{P}_1} P(\phi=0) \leq \beta$, with $\mathcal{P}_1 \subset \mathcal{P}\setminus\mathcal{P}_0$ as large as possible.

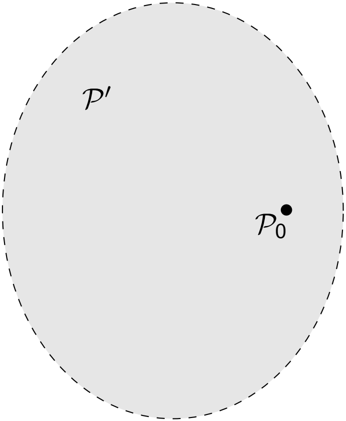
✘ In general, if $\alpha+\beta<1$, $\mathcal{P}_1$ can not be equal to $\mathcal{P}\setminus\mathcal{P}_0$!

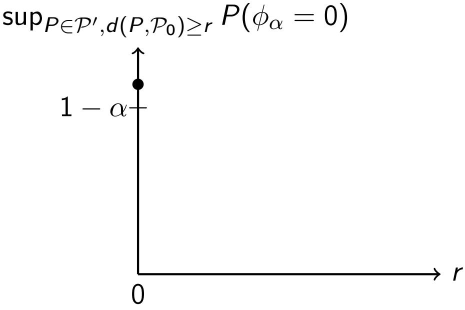
➟ $\mathcal{P}_1 = \{P\in\mathcal{P}',\ d(P,\mathcal{P}_0)\geq r\}$, with $r$ as small as possible,

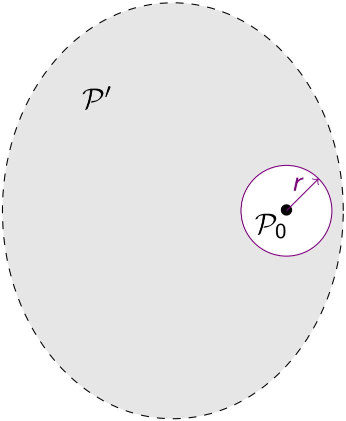
for some distance $d$ on $\mathcal{P}$, and (realistic ?) restricted class of probability distributions $\mathcal{P}'\subset\mathcal{P}$.

Let $\phi_\alpha$ be a level $\alpha$ test of $(H_0)$ against $(H_1)$.
The **uniform separation rate** of $\phi_\alpha$ over $\mathcal{P}'$ is defined as

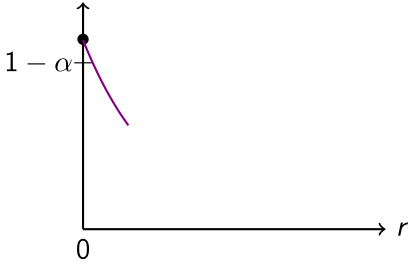
$$\mathrm{SR}_d^\beta(\phi_\alpha,\mathcal{P}') = \inf\left\{r>0,\ \sup_{P\in\mathcal{P}',d(P,\mathcal{P}_0)\geq r} P(\phi_\alpha=0)\leq\beta\right\}.$$

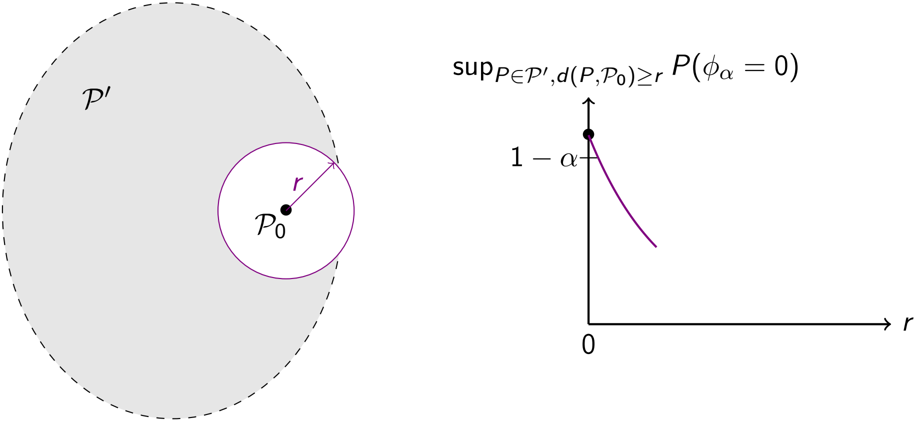

$\mathcal{P}'$
$\mathcal{P}_0$


$\sup_{P \in \mathcal{P}', d(P, \mathcal{P}_0) \geq r} P(\phi_\alpha = 0)$
$1 - \alpha$
$0$
$r$




$\sup_{P\in\mathcal{P}',d(P,\mathcal{P}_0)\geq r} P(\phi_\alpha = 0)$

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$r$
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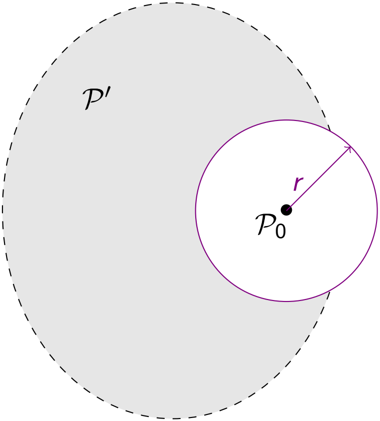
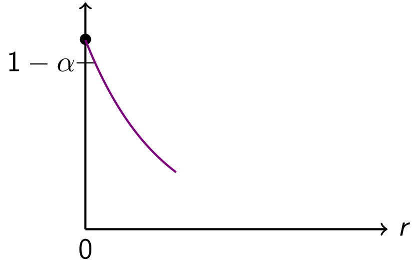
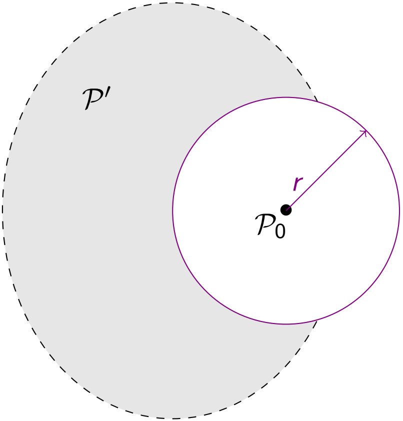
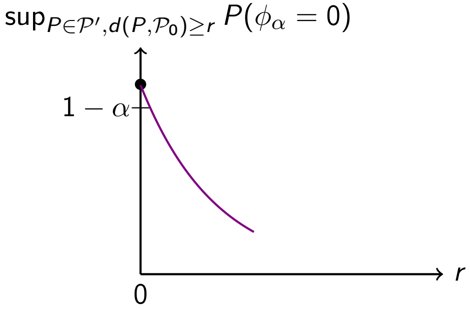



$\sup_{P \in \mathcal{P}', d(P,\mathcal{P}_0) \geq r} P(\phi_\alpha = 0)$




$\mathcal{P}'$
$r$
$\mathcal{P}_0$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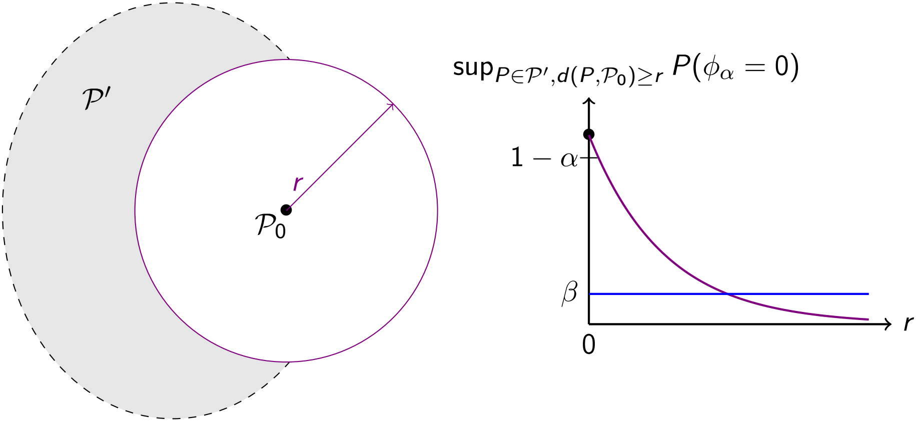
$1-\alpha$
0
$r$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$\mathcal{P}_0$
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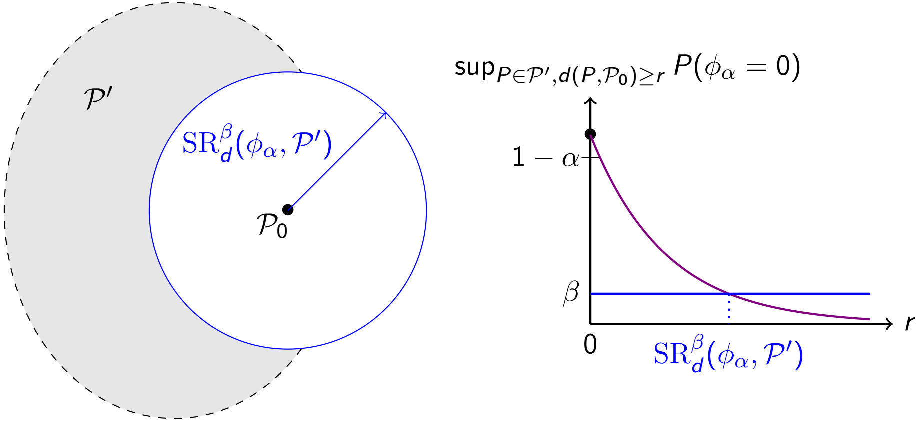
$1-\alpha$
$\beta$
$0$
$r$


$\mathcal{P}'$
$\mathrm{SR}_d^\beta(\phi_\alpha, \mathcal{P}')$
$\mathcal{P}_0$
$\sup_{P\in\mathcal{P}', d(P,\mathcal{P}_0)\geq r} P(\phi_\alpha = 0)$
$1-\alpha$
$\beta$
$0$
$r$
$\mathrm{SR}_d^\beta(\phi_\alpha, \mathcal{P}')$

# Single tests of single null hypotheses

## Nonasymptotic minimax testing

A second kind error related criterion which allows to:

- Compare two level $\alpha$ tests
- See whether a level $\alpha$ test is optimal over $\mathcal{P}'$, in the following minimax sense.

The **minimax separation rate** over $\mathcal{P}'$ is defined by

$$m\mathrm{SR}_d^{\alpha,\beta}(\mathcal{P}') = \inf_{\{\phi_\alpha \text{ of level } \alpha\}} \mathrm{SR}_d^{\beta}(\phi_\alpha, \mathcal{P}').$$

A level $\alpha$ test $\phi_\alpha$ is **minimax** over $\mathcal{P}'$, if

$$\mathrm{SR}_d^{\beta}(\phi_\alpha, \mathcal{P}') \leq C(\alpha, \beta)\ m\mathrm{SR}_d^{\alpha,\beta}(\mathcal{P}').$$

➠ Parallel between the minimax hypothesis testing theory and the minimax estimation theory

# Single tests of single null hypotheses

## Nonasymptotic minimax testing: example in the density model

**Density model** | $\mathbf{X} = (X_1, \ldots, X_n)$ is a sample of $n$ i.i.d. random variables with distribution $P_f$ of density $f$ with respect to the Lebesgue measure $\lambda$ on $\mathbb{X} = \mathbb{R}$, $\mathcal{P} = \{P_f,\ f \in \mathbb{L}_2(\mathbb{R}, \lambda)\}$.

**Goodness-of-fit test**: given a density $f_0 \in \mathbb{L}_2(\mathbb{R}, \lambda)$,

$$(H_0)\ f = f_0 \Leftrightarrow P_f \in \mathcal{P}_0 = \{P_{f_0}\} \text{ against } (H_1)\ f \neq f_0 \Leftrightarrow P_f \notin \mathcal{P}_0 = \{P_{f_0}\}$$

**Minimax separation rate**: $d_2(P_f, P_g) = \|f - g\|_2$, $\mathcal{B}_{s,\infty,\infty}(R)$ Hölder ball,

$$m\mathrm{SR}_{d_2}^{\alpha,\beta}(\{P_f,\ f \in \mathcal{B}_{s,\infty,\infty}(R)\}) \approx n^{-2s/(4s+1)}$$

✏ Ingster (1993), Pouet (2002)

# Single tests of single null hypotheses

## Nonasymptotic minimax testing: example in the density model

$S_m = \langle b_{m,k},\ k \in \mathbb{Z} \rangle$, with $b_{m,k} = \sqrt{m}\mathbb{1}_{[k/m,(k+1)/m)}$ for $m \in \mathbb{N}\backslash\{0\}$, $\Pi_{S_m}$ orthogonal projection onto $S_m$ w.r.t. $\langle .,. \rangle_2$

$(H_{0,m})$ $P_f \in \mathcal{P}_{0,m}$, with $\mathcal{P}_{0,m} = \left\{ P_f,\ \Pi_{S_m}(f - f_0) = 0 \right\} \supset \mathcal{P}_0$.

**Single test**: $\phi_{m,\alpha} = \mathbb{1}_{\left\{ T_m > F_m^{-1}(1-\alpha) \right\}}$, with

$T_m = \frac{1}{n(n-1)} \sum_{k \in \mathbb{Z}} \sum_{i \neq j = 1}^{n} b_{m,k}(X_i) b_{m,k}(X_j) + \|f_0\|_2^2 - \frac{2}{n} \sum_{i=1}^{n} f_0(X_i)$

estimating $\|\Pi_{S_m}(f - f_0)\|_2^2$, $F_m =$ c.d.f. of $T_m$ under $(H_0)$

$\phi_{m,\alpha}$ is a level $\alpha$ test such that $P_f(\phi_{m,\alpha} = 0) \leq \beta$ as soon as

$$d_2^2(P_f, \mathcal{P}_0) > (1 + \varepsilon) \left\{ \|f - \Pi_{S_m}(f)\|_2^2 + C \left( \frac{\sqrt{m \ln(1/\alpha)}}{n} + \frac{m}{n^2} \right) \right\}.$$

✏ Fromont, Laurent, *Ann. Stat.* (2006)

Tools: concentration inequalities ($U$ statistics of order 2, linear statistics)

# Single tests of single null hypotheses

## Nonasymptotic minimax testing: example in the density model

**Bias term:** for $s \in (0,1]$, $f \in \mathcal{B}_{s,\infty,\infty}(R) \Rightarrow \|f - \Pi_{S_m}(f)\|^2 \leq C(s)R^2 m^{-2s}$

**Minimax test:** Take $m$ such that $R^2 m^{-2s} \simeq \sqrt{m}/n \Leftrightarrow m \simeq (R^2 n)^{2/(4s+1)}$.

For $n$ large,
$\mathrm{SR}_{d_2}^{\beta}(\phi_{m,\alpha}, \{P_f,\ f \in \mathcal{B}_{s,\infty,\infty}(R) \cap \mathbb{L}_\infty(R')\}) \leq C(s,\alpha,\beta,R')R^{\frac{1}{4s+1}} n^{\frac{-2s}{4s+1}}$.

✘ Problem: the test depends on $s$! A priori **realistic** choice of $\mathcal{B}_{s,\infty,\infty}(R)$ ?

➟ Test which does not depend on $s$ but which is minimax or nearly minimax over the class $\{P_f,\ f \in \mathcal{B}_{s,\infty,\infty}(R) \cap \mathbb{L}_\infty(R')\}$ for every $s$?

A level $\alpha$ test $\phi_\alpha$ is **minimax adaptive** over a collection of classes $\mathcal{P}'$, if it is minimax or nearly minimax over all the classes $\mathcal{P}'$ in the collection.

➟ Aggregation of tests

# Aggregated tests

- Collection of subsets of $\mathcal{P}$: $\{\mathcal{P}_{0,m},\ m \in \mathcal{M}\}$, $\mathcal{P}_0 \subset \cap_{m\in\mathcal{M}}\mathcal{P}_{0,m}$
- Collection of hypotheses: $\{(H_{0,m}),\ m \in \mathcal{M}\}$, $(H_{0,m})\ P \in \mathcal{P}_{0,m}$
- Collection of tests: $\Phi_\alpha = \left\{\phi_{m,\alpha} = \mathbb{1}_{\{T_m > q_m(1-\alpha)\}},\ m \in \mathcal{M}\right\}$ with $\sup_{P\in\mathcal{P}_0} P(\phi_{m,\alpha} = 1) \leq \alpha$
- Collection of individual levels: $U_\alpha = \{u_{m,\alpha},\ m \in \mathcal{M}\}$

The **aggregated test** based on the collections $\Phi_\alpha$ and $U_\alpha$ is defined as

$$\bar{\Phi}_\alpha = \sup_{m\in\mathcal{M}} \phi_{m,u_{m,\alpha}} = \sup_{m\in\mathcal{M}} \mathbb{1}_{\{T_m > q_m(1-u_{m,\alpha})\}}.$$

➟ Reject $(H_0)$ if at least one $(H_{0,m})$ is rejected with $\phi_{m,u_{m,\alpha}}$

# Aggregated tests

Two concerns: level control + minimax adaptivity

- Minimax adaptivity: choice of $T_m$ (minimax single tests)
- Level control: choice of $q_m(1 - u_{m,\alpha})$

Four different cases can be distinguished ($Z$ is a statistic depending on $\mathbf{X}$).

**Notation:**

$$\begin{aligned} \mathcal{L}_{(H_0)}(T) &= \text{distribution of } T \text{ given } Z, \\ \mathcal{L}_{(H_0)}(T|Z) &= \text{conditional distribution of } T \text{ given } Z \text{ under } (H_0), \\ \mathcal{L}(T|Z) &= \text{conditional distribution of } T \text{ given } Z \end{aligned}$$

[$KD$] (Known Distr.) $\mathcal{L}_{(H_0)}(T_m)$ is known (parameter free)

[$UD$] (Unknown Distr.) →

- [UD1] $\mathcal{L}_{(H_0)}(T_m|Z)$ is known
- [$UD2$] $\exists T_m^*,\ \mathcal{L}(T_m^*|Z) = \mathcal{L}_{(H_0)}(T_m|Z)$
- [$UD3$] $\exists T_m^*,\ \mathcal{L}_{(H_0)}(T_m^*|Z) = \mathcal{L}_{(H_0)}(T_m|Z)$

# Aggregated tests: goodness-of-fit

## In the density model ([KD])

$S_m = \langle b_{m,k},\ k \in \mathbb{Z} \rangle$, with $b_{m,k} = \sqrt{m}\mathbb{1}_{[k/m,(k+1)/m)}$ for $m \in \mathbb{N} \setminus \{0\}$

- Collection of subsets of $\mathcal{P}$: $\{\mathcal{P}_{0,m} = \{P_f,\ \Pi_{S_m}(f - f_0) = 0\},\ m \in \mathcal{M}\}$
- Collection of hypotheses: $\{(H_{0,m})\ P_f \in \mathcal{P}_{0,m},\ m \in \mathcal{M}\}$
- Collection of tests: $\left\{\phi_{m,\alpha} = \mathbb{1}_{\left\{T_m > F_m^{-1}(1-\alpha)\right\}}, m \in \mathcal{M}\right\}$,
- Collection of individual levels: $\{u_{m,\alpha},\ m \in \mathcal{M}\}$ ?

**Bonferroni choice:** $u_{m,\alpha} = \alpha / \#\mathcal{M}$

$$\bar{\Phi}_\alpha^{Bonf} = \sup_{m \in \mathcal{M}} \phi_{m,\alpha/\#\mathcal{M}} = \sup_{m \in \mathcal{M}} \mathbb{1}_{\left\{T_m > F_m^{-1}(1-\alpha/\#\mathcal{M})\right\}}$$

**FL choice:** $u_{m,\alpha} = u_\alpha = \sup \left\{u,\ \mathbb{P}_{(H_0)}\left(\exists m \in \mathcal{M},\ T_m > F_m^{-1}(1-u)\right) \leq \alpha\right\}$

$$\bar{\Phi}_\alpha^{FL} = \sup_{m \in \mathcal{M}} \phi_{m,u_\alpha} = \sup_{m \in \mathcal{M}} \mathbb{1}_{\left\{T_m > F_m^{-1}(1-u_\alpha)\right\}}$$

$\bar{\Phi}_\alpha^{Bonf}$ and $\bar{\Phi}_\alpha^{FL}$ are both of level $\alpha$, and $\bar{\Phi}_\alpha^{FL}$ is less conservative than $\bar{\Phi}_\alpha^{Bonf}$

# Aggregated tests: goodness-of-fit

## In the density model ([KD])

$P_f(\bar{\Phi}_\alpha^{FL} = 0) \leq \beta$ as soon as

$$d_2^2(P_f, \mathcal{P}_0) > (1+\varepsilon) \inf_{m \in \mathcal{M}} \left\{ \|f - \Pi_{S_m}(f)\|_2^2 + C \left( \frac{\sqrt{m \ln(\#\mathcal{M}/\alpha)}}{n} + \frac{m}{n^2} \right) \right\}$$

✏ Fromont, Laurent, *Ann. Stat.* (2006)

Taking $\mathcal{M}$ with $\#\mathcal{M} \simeq \ln n$ ➟ loss in $\sqrt{\ln \ln n}$

For $n$ large enough, $s \in (0,1]$, $\mathcal{M} = \left\{ 2^J,\ 0 \leq J \leq \log_2 \left( n^2/(\ln \ln n)^3 \right) \right\}$,

$$SR_{d_2}^\beta \left( \bar{\Phi}_\alpha^{FL}, \left\{ P_f, f \in \mathcal{B}_{s,\infty,\infty}(R) \cap \mathbb{L}_\infty(R') \right\} \right) \leq C\ R^{\frac{1}{4s+1}} \left( \sqrt{\ln \ln n}/n \right)^{\frac{2s}{4s+1}}$$

- ➟ $\bar{\Phi}_\alpha^{FL}$ is minimax adaptive with an unavoidable (✏ Ingster (2000)) loss the order of a $\sqrt{\ln \ln n}$ factor.
- ➟ Extension to test that $f$ belongs to a translation/scale family: similar results but with a loss of the order of a $\sqrt{\ln n}$ factor

# Aggregated tests: goodness-of-fit

## In the Poisson model ([UD1])

**Poisson model** | $\mathbf{X} = \{X_1, \ldots, X_{N_\mathbf{X}}\}$ is a Poisson process on $\mathbb{X} = [0,1]$, with intensity $f$ w.r.t. $d\mu = nd\lambda$, whose distribution is denoted by $P_f$, $\mathcal{P} = \{P_f,\ f \in \mathbb{L}_2(\mathbb{R}, \lambda)\}$.

**Homogeneity test**

$(H_0)\quad P_f \in \mathcal{P}_0 = \{P_f,\ f \text{ constant}\}$ against $(H_1)\quad P_f \notin \mathcal{P}_0$

**Motivation**: Detecting abnormal behaviors on the DNA sequence

➡ Detecting alternative intensities with localized spikes

**Minimax separation rate?** $d_2(P_f, P_g) = \|f - g\|_2$ (w.r.t. $\lambda$),

$\mathcal{B}_{s,2,\infty}(R)$ (strong) Besov body, $w\mathcal{B}_{s'}(R')$ weak Besov body

defined from the Haar basis $\{\varphi_0,\ \psi_{(j,k)},\ j \in \mathbb{N},\ k \in \{0, \ldots, 2^j - 1\}\}$.

$$\mathcal{B}_{s,2,\infty}(R) = \left\{ f,\ \forall j \in \mathbb{N},\ \sum_{k=0}^{2^j-1} \langle f, \psi_{(j,k)} \rangle_2^2 \leq R^2 2^{-2js} \right\}$$

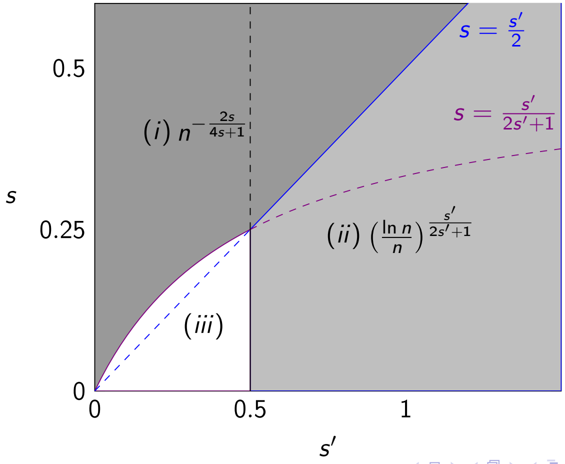
$$w\mathcal{B}_{s'}(R') = \left\{ f, \forall t > 0, \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} \langle f, \psi_{(j,k)} \rangle_2^2 \, \mathbb{1}_{\langle f, \psi_{(j,k)} \rangle_2^2 \leq t} \leq R'^2 t^{\frac{2s'}{2s'+1}} \right\}$$

# Aggregated tests: goodness-of-fit

## In the Poisson model ([UD1])

$$m\mathrm{SR}_{d_2}^{\alpha,\beta}\left(\{P_f, f\in\mathcal{B}_{s,2,\infty}(R)\cap w\mathcal{B}_{s'}(R')\cap\mathbb{L}_\infty(R'')\}\right)$$

✏ Fromont, Laurent, Reynaud-Bouret, *Ann. IHP* (2011)



# Aggregated tests: goodness-of-fit

## In the Poisson model ([UD1])

### Aggregated homogeneity tests

$S_m = \langle \varphi_0, \psi_{(j,k)}, (j,k) \in \mathcal{L}_m \rangle$, with
$\mathcal{L}_m \subset \{(j,k), j \in \mathbb{N}, k \in \{1, \ldots, 2^j - 1\}\}$, $m \in \mathcal{M}$

- Collection of subsets of $\mathcal{P}$: $\{\mathcal{P}_{0,m} = \{P_f, \Pi_{S_m}(f) \text{ is constant}\}, m \in \mathcal{M}\}$
- Collection of hypotheses: $\{(H_{0,m})\ \ P \in \mathcal{P}_{0,m},\ \ m \in \mathcal{M}\}$
- Collection of single tests: $\left\{\phi_{m,\alpha} = \mathbb{1}_{\left\{T_m > q_m^{N_{\mathbf{X}}}(1-\alpha)\right\}},\ m \in \mathcal{M}\right\}$, with

$$T_m = \frac{1}{n^2} \sum_{(j,k)\in\mathcal{L}_m} \sum_{i \neq i'=1}^{N_{\mathbf{X}}} \psi_{(j,k)}(X_i)\psi_{(j,k)}(X'_i) \overset{\text{est.}}{\rightarrow} \|\Pi_{\langle \psi_{(j,k)},\ (j,k)\in\mathcal{L}_m\rangle}(f)\|_2^2$$

$q_m^{n_0}$ quantile function of $\mathcal{L}_{(H_0)}(T_m | N_{\mathbf{X}} = n_0)$, which is known since

$\mathcal{L}_{(H_0)}(T_m | N_{\mathbf{X}} = n_0) = \mathcal{L}\left(\frac{1}{n^2} \sum_{(j,k)\in\mathcal{L}_m} \sum_{i \neq i'=1}^{n_0} \psi_{(j,k)}(U_i)\psi_{(j,k)}(U'_i)\right)$, with $(U_1, \ldots, U_{n_0})$ i.i.d. uniformly distributed (case $[UD1]$ with $Z = N_{\mathbf{X}}$)

# Aggregated tests: goodness-of-fit

## In the Poisson model ([UD1])

- Collection of individual levels: $\{ u_{m,\alpha}, \ m \in \mathcal{M} \}$ ?

**FLR choice:** $u_{m,\alpha} = u_{m,\alpha}^{N_{\mathbf{X}}}$, with

$u_{m,\alpha}^{n_0} = w_m \sup \left\{ u, \mathbb{P}_{(H_0)} \left( \exists m \in \mathcal{M}, T_m > q_m^{(n_0)}(1 - w_m u) \middle| N_{\mathbf{X}} = n_0 \right) \leq \alpha \right\}$,

$(w_m)_{m \in \mathcal{M}}$ positive weights such that $\sum_{m \in \mathcal{M}} w_m \leq 1$

$$\bar{\Phi}_\alpha^{FLR} = \sup_{m \in \mathcal{M}} \phi_{m, u_{m,\alpha}} = \sup_{m \in \mathcal{M}} \mathbb{1}_{\left\{ T_m > q_m^{N_{\mathbf{X}}} \left( 1 - u_{m,\alpha}^{N_{\mathbf{X}}} \right) \right\}}$$

$D_m = \dim(S_m)$, $E_m = \sum_{j/(j,k) \in \mathcal{L}_m} 2^j$.
Then $\bar{\Phi}_\alpha^{FLR}$ is of level $\alpha$ and $P_f(\bar{\Phi}_\alpha^{FLR} = 0) \leq \beta$ as soon as
$d_2^2(P_f, \mathcal{P}_0) >$
$\inf_{m \in \mathcal{M}} \left\{ \|f - \Pi_{S_m}(f)\|_2^2 + C \left( \frac{\sqrt{D_m \ln(1/(w_m \alpha))}}{n} + \frac{\ln(1/(w_m \alpha))}{n} + \frac{E_m \ln^2(1/(w_m \alpha))}{n^2} \right) \right\}$

**Probabilistic tools:** concentration inequalities ($U$ statistics of order 2)

# Aggregated tests: goodness-of-fit

## In the Poisson model ([UD1])

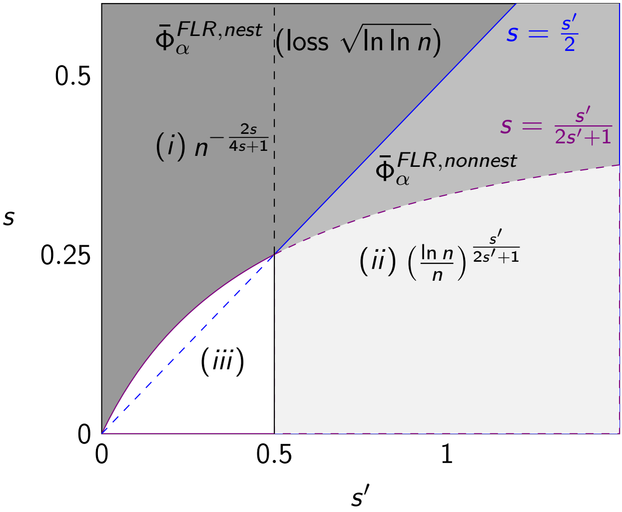
$D_m = \dim(S_m)$, $E_m = \sum_{j/(j,k)\in\mathcal{L}_m} 2^j$.
Then $\bar\Phi_\alpha^{FLR}$ is of level $\alpha$ and $P_f(\bar\Phi_\alpha^{FLR} = 0) \leq \beta$ as soon as
$d_2^2(P_f, \mathcal{P}_0) >$
$$\inf_{m\in\mathcal{M}} \left\{ \|f - \Pi_{S_m}(f)\|_2^2 + C\left( \frac{\sqrt{D_m \ln(1/(w_m\alpha))}}{n} + \frac{\ln(1/(w_m\alpha))}{n} + \frac{E_m \ln^2(1/(w_m\alpha))}{n^2} \right) \right\}$$

Which choice for $\{S_m,\ m\in\mathcal{M}\}$ and $(w_m)_{m\in\mathcal{M}}$ ?

- Classical collection of nested spaces: allows to detect intensities in $\mathcal{B}_{s,2,\infty}(R)$, $E_m \simeq D_m \Rightarrow w_m = 1/\#\mathcal{M}$ possible $\Rightarrow \bar\Phi_\alpha^{FLR,nest}$ minimax adaptive with a loss $\sim \sqrt{\ln\ln n}$ factor.
- Need for a richer collection of nonnested spaces to detect intensities in $\mathcal{B}_{s,2,\infty}(R) \cap w\mathcal{B}_{s'}(R')$ with $s \geq s'/(2s'+1)$, $s' > 1/2 \Rightarrow \#\mathcal{M}$ large $\Rightarrow$ other choice for $w_m$ $\Rightarrow \bar\Phi_\alpha^{FLR,nonnest}$ minimax adaptive without any loss

# Aggregated tests: goodness-of-fit

## In the Poisson model ([UD1])



# Aggregated tests: two-sample problems

## In the Poisson model ([*UD*2])

**Poisson model** | $\mathbf{X} = (\mathbf{X}^1, \mathbf{X}^2)$ is a pair of independent Poisson processes, observed on $\mathbb{X} \subset \mathbb{R}^d$, with resp. intensities $f_1$ and $f_2$ (in $\mathbb{L}_1(\mathbb{X}, \lambda) \cap \mathbb{L}_\infty(\mathbb{X})$), w.r.t $d\mu = nd\lambda$.
$P_{f_1,f_2}$ = joint distribution of $\mathbf{X} = (\mathbf{X}^1, \mathbf{X}^2)$.

$$(H_0)\ \ f_1 = f_2 \Leftrightarrow P_{(f_1,f_2)} \in \mathcal{P}_0 = \left\{ P_{(f_1,f_2)},\ f_1 = f_2 \right\} \text{ against } (H_1)\ \ P_{(f_1,f_2)} \notin \mathcal{P}_0$$

### Motivations

- Differential analysis of replication origins peaks
- Spatial representativeness of services in public statistics

### Notations

$\mathbf{X}^1 = \{X_1^1, \ldots, X_{N_1}^1\}$, $\mathbf{X}^2 = \{X_1^2, \ldots, X_{N_2}^2\}$ ($N_1, N_2$ random),
$\bar{\mathbf{X}} = \mathbf{X}^1 \cup \mathbf{X}^2 = \{X_1, \ldots, X_N\}$ with $N = N_1 + N_2$.

# Aggregated tests: two-sample problems

## In the Poisson model ([$UD$2])

### Single kernel based test

Considering as above a subspace $S_m = \langle b_l, l \in \mathcal{L}_m \rangle$ (orthonormal basis) of $\mathbb{L}_2(\mathbb{X}, \lambda)$, a natural idea is to introduce
$(H_{0,m})$ $P_f \in \mathcal{P}_{0,m}$, with $\mathcal{P}_{0,m} = \{ P_f, \ \Pi_{S_m}(f_1 - f_2) = 0 \} \supset \mathcal{P}_0$.

Unbiased estimator of $n^2 \|\Pi_{S_m}(f_1 - f_2)\|_2^2$:

$$T_m = \sum_{i \neq j = 1}^{N} \Big( \sum_{l \in \mathcal{L}_m} b_l(X_i) b_l(X_j) \Big) \varepsilon_i^0 \varepsilon_j^0, \text{ where } \left| \begin{array}{rl} \varepsilon_i^0 = & 1 \text{ if } X_i \in \mathbf{X}^1, \\ \varepsilon_i^0 = & -1 \text{ if } X_i \in \mathbf{X}^2. \end{array} \right.$$

➡ Generalization to $T_m = \sum_{i \neq j=1}^{N} K_m(X_i, X_j) \varepsilon_i^0 \varepsilon_j^0$, where $K_m$ is a symmetric kernel s. t. $\int K_m^2(x, x')(f_1 + f_2)(x)(f_1 + f_2)(x') d\nu(x) d\nu(x') \leq D_m$

➡ Unbiased estimator of $n^2 \langle K_m[f_1 - f_2], f_1 - f_2 \rangle_2$ with $K_m[f](x) = \langle K_m(., x), f \rangle_2$

# Aggregated tests: two-sample problems

In the Poisson model ([$UD2$])

## Possible choices for the kernel

- [$PK$] projection kernel $K_m(x,x') = \sum_{l \in \mathcal{L}_m} b_l(x) b_l(x')$,
  $\langle K_m[f_1 - f_2], f_1 - f_2 \rangle_2 = \|\Pi_{S_m}(f_1 - f_2)\|_2^2$
- [$AK$] approximation kernel $K_m(x,x') = k_m\left(\frac{x_1 - x_1'}{h_1}, \ldots, \frac{x_d - x_d'}{h_d}\right) / \prod_{i=1\ldots d} h_i$,
  $\langle K_m[f_1 - f_2], f_1 - f_2 \rangle_2 = \langle k_m * (f_1 - f_2), f_1 - f_2 \rangle_2 / \prod_{i=1\ldots d} h_i$
- [$RK$] reproducing kernel $K_m(x,x') = \langle \theta_{K_m}(x), \theta_{K_m}(x') \rangle_{\mathcal{H}_{K_m}}$, $\theta_{K_m}$ and $\mathcal{H}_{K_m}$ feature function and RKHS space,
  $\langle K_m[f_1 - f_2], f_1 - f_2 \rangle_2 = \|K_m[f_1] - K_m[f_2]\|^2_{\mathcal{H}_{K_m}}$, $K_m[f_1]$ and $K_m[f_2]$ mean embeddings of $f_1$ and $f_2$ in the RKHS if they are densities.

**Single test**: $\phi_{m,\alpha} = \mathbb{1}_{\{T_m > q_m(1-\alpha)\}}$, $q_m$ to define

✘ Problem: the distribution of $T_m$ is not free from $f_1 = f_2$ under $(H_0)$ !

➟ Wild bootstrap approach ✏ Fromont, Laurent, Reynaud-Bouret, *Ann. Stat.* (2013)

# Aggregated tests: two-sample problems

In the Poisson model ([$UD2$])

## Wild bootstrap approach in the density model

- Classical Efron's bootstrap
  - Empirical process: $(P_n - P)(h) \rightarrow (P_n^* - P_n)(h) = \frac{1}{n}\sum_{i=1}^n (M_{n,i} - 1)h(X_i)$
    ✏ Giné, Zinn (1990,1992)
  - Degenerate $U$-statistics: $U_n(h) = \frac{1}{n(n-1)}\sum_{i\neq j} h(X_i, X_j)$
    $\rightarrow U_n^*(h) = \frac{1}{n(n-1)}\sum_{i\neq j} h(X_i, X_j)(M_{n,i} - 1)(M_{n,j} - 1)$ ✏ Arcones, Giné (1992)
- Wild bootstrap based on i.i.d. Rademacher variables $(\varepsilon_1, \ldots, \varepsilon_n)$
  - Empirical process: $(P_n - P)(h) \rightarrow (P_n^* - P_n)(h) = \frac{1}{n}\sum_{i=1}^n \varepsilon_i h(X_i)$
    ✏ Mammen (1992)
    ✏ Fromont, *Mach. Learn.* (2007) for nonasymptotic results
  - Degenerate $U$-statistics: $U_n(h) = \frac{1}{n(n-1)}\sum_{i\neq j} h(X_i, X_j)$
    $\rightarrow U_n^*(h) = \frac{1}{n(n-1)}\sum_{i\neq j} h(X_i, X_j)\varepsilon_i\varepsilon_j$ ✏ Dehling Mikosch (1994)

# Aggregated tests: two-sample problems

## In the Poisson model ([*UD*2])

**Wild bootstrap approach in the Poisson model**

$T_m^* = \sum_{i \neq j} K_m(X_i, X_j)\varepsilon_i\varepsilon_j \quad \Rightarrow \quad \mathcal{L}(T_m^* | \bar{\mathbf{X}}) = \mathcal{L}_{(H_0)}(T_m | \bar{\mathbf{X}})$ [*UD*2]

$q_m = q_m^{\bar{\mathbf{X}}} =$ quantile function of $\mathcal{L}(T_m^* | \bar{\mathbf{X}})$ (Monte Carlo)

$\phi_{m,\alpha} = \mathbb{1}_{\{T_m > q_m^{\bar{\mathbf{X}}}(1-\alpha)\}}$ is of level $\alpha$,
even when $q_m^{\bar{\mathbf{X}}}(1-\alpha)$ is approximated by a Monte Carlo method!
✏ Fromont, Laurent, Reynaud-Bouret, *Ann. Stat.* (2013) / Fromont, HDR (2015)

Tool: key exchangeability lemma ✏ Romano, Wolf (2005)

# Aggregated tests: two-sample problems

## In the Poisson model ([$UD2$])

### Aggregated two-sample tests

- Collection of subsets of $\mathcal{P}$:
  $\{\mathcal{P}_{0,m} = \{ P_{f_1,f_2},\ \langle K_m[f_1 - f_2], f_1 - f_2\rangle_2 = 0 \}, m \in \mathcal{M}\}$
- Collection of hypotheses: $\{ (H_{0,m})\ \ P \in \mathcal{P}_{0,m},\ \ m \in \mathcal{M} \}$
- Collection of single tests: $\left\{ \phi_{m,\alpha} = \mathbb{1}_{\left\{ T_m > q_m^{\bar{\mathbf{X}}}(1-\alpha) \right\}},\ \ m \in \mathcal{M} \right\}$
- Collection of individual levels: $\{ u_{m,\alpha},\ \ m \in \mathcal{M} \}$ ?

**FLR choice:** $u_{m,\alpha} = u_{m,\alpha}^{\bar{\mathbf{X}}}$, with

$$u_{m,\alpha}^{\bar{\mathbf{X}}} = w_m \sup \left\{ u, \mathbb{P}_{(H_0)} \left( \exists m \in \mathcal{M}, T_m^* > q_m^{\bar{\mathbf{X}}}(1 - w_m u) \middle| \bar{\mathbf{X}} \right) \leq \alpha \right\},$$

$$\bar{\Phi}_\alpha^{FLR} = \sup_{m \in \mathcal{M}} \phi_{m, u_{m,\alpha}} = \sup_{m \in \mathcal{M}} \mathbb{1}_{\left\{ T_m > q_m^{\bar{\mathbf{X}}}\left(1 - u_{m,\alpha}^{\bar{\mathbf{X}}}\right) \right\}}$$

# Aggregated tests: two-sample problems

In the Poisson model ([$UD2$])

## Oracle type result

The test $\bar{\Phi}_{\alpha}^{FLR}$ is of level $\alpha$ and $P_{f_1,f_2}(\bar{\Phi}_{\alpha}^{FLR}=0)\leq\beta$, as soon as
$\|f_1-f_2\|_2^2\geq$
$\inf_{m\in\mathcal{M}}\inf_{r>0}\left\{\left\|(f_1-f_2)-r^{-1}K_m[f_1-f_2]\right\|_2^2+C\left(\frac{\sqrt{D_m}\ \ln(1/(w_m\alpha))}{rn}\right)\right\}.$

Fromont, Laurent, Reynaud-Bouret, *Ann. Stat.* (2013)

Tools: concentration inequalities & exponential inequalities for Rademacher chaos

**Minimax adaptivity properties** over:

- $\{P_{f_1,f_2},\ (f_1-f_2)\in\mathcal{B}_{s,2,\infty}(R)\cap w\mathcal{B}_{s'}(R'),\ f_1,f_2\in\mathbb{L}_\infty(R'')\}$ (loss $\sim$ $(\ln\ln n)$ in the case $(i)$, no loss in the case $(ii)$)
- subsets based on $d$ dim. Sobolev and anisotropic Nikol'skii-Besov balls (loss $\sim$ $(\ln\ln n)$)

➡ Parametric rate for the single tests based on characteristic kernels for the weak distance $\|K_m[f_1]-K_m[f_2]\|_{\mathcal{H}_{K_m}}$ $\Rightarrow$ choice of the distance?

# Aggregated tests: two-sample problems

## In the density model ([UD3])

**Density model** | $\mathbf{X} = (\mathbf{X}^1, \mathbf{X}^2)$ is a pair of independent sets of i.i.d. random variables, with respective densities $f_1$ and $f_2$, w.r.t. $\lambda$.

$(H_0)\ f_1 = f_2 \Leftrightarrow P_{(f_1,f_2)} \in \mathcal{P}_0 = \left\{ P_{(f_1,f_2)},\ f_1 = f_2 \right\}$ against $(H_1)\ P_{(f_1,f_2)} \notin \mathcal{P}_0$

**Aggregated tests based on kernels as in the Poisson process model**

$T_m = \sum_{i \neq j=1}^{N} K_m(X_i, X_j)\varepsilon_i^0 \varepsilon_j^0$, where if $c_{N_1,N_2} = 1/(N_1 N_2(N_1 + N_2 + 2))$,

$\varepsilon_i^0 = a_{N_1,N_2} = (1/(N_1(N_1 - 1)) - c_{N_1,N_2})^{1/2}$ if $X_i \in \mathbf{X}^1$,

$\varepsilon_i^0 = b_{N_1,N_2} = -a_{N_2,N_1}$ if $X_i \in \mathbf{X}^2$.

➡ $T_m + c_{N_1,N_2} \sum_{i \neq j=1}^{N} K_m(X_i, X_j)$ unbiased estimator of $\langle K_m[f_1 - f_2], f_1 - f_2 \rangle_2$

✏ Fromont, Laurent, Lerasle, Reynaud-Bouret *JMLR Proc., COLT* (2012)

**Another kind of possible (nonsymmetric) kernel** based on $k_m$ nearest neighbors: $K_m(x, x') = \mathbb{1}_{\{x'\, k_m\text{-nn of } x\}}$, with other marks

➡ less complex collections $\Rightarrow$ possible extension to functional data

✏ Fromont, Tuleau, *JMLR Proc., COLT* (2006) / Fromont, Tuleau (2015)

# Aggregated tests: two-sample problems

## In the density model ([UD3])

### Bootstrap approach

Wild bootstrap ➟ asymptotically valid in the density model, but

Permutation ➟ "exact" bootstrap approach in the density model

$$\varepsilon_i = a_{N_1,N_2} \text{ if } \Pi_N(i) \in \{1, \ldots, N_1\},$$
$$\varepsilon_i = b_{N_1,N_2} \text{ if } \Pi_N(i) \in \{N_1+1, \ldots, N\},$$

$\Pi_N$ random permutation uniformly distributed on $\mathfrak{S}_N$.

$$T_m^* = \sum_{i \neq j} K_m(X_i, X_j)\varepsilon_i\varepsilon_j \quad \Rightarrow \quad \mathcal{L}_{(H_0)}(T_m^*|\bar{\mathbf{X}}) = \mathcal{L}_{(H_0)}(T_m|\bar{\mathbf{X}}) \quad [UD3]$$

$q_m = q_m^{\bar{\mathbf{X}}} =$ quantile function of $\mathcal{L}(T_m^*|\bar{\mathbf{X}})$ (Monte Carlo)

$\phi_{m,\alpha} = \mathbb{1}_{\{T_m > q_m^{\bar{\mathbf{X}}}(1-\alpha)\}}$ is of level $\alpha$,

even when $q_m^{\bar{\mathbf{X}}}(1-\alpha)$ is approximated by a Monte Carlo method!

# Multiple tests

## Parallel between aggregated tests and multiple tests

Collection of hypotheses: $\{(H_{0,m})\ P \in \mathcal{P}_{0,m},\ m \in \mathcal{M}\}$

**Aggregated tests in the case $[KD]$**

Testing $(H_0)$ $P \in \mathcal{P}_0 \subset \cap_{m\in\mathcal{M}} \mathcal{P}_{0,m}$ against $(H_1)$ $P \notin \mathcal{P}_0$

Minimax adaptive level $\alpha$ aggregated tests: $\bar{\Phi}_\alpha^{Bonf}$, $\bar{\Phi}_\alpha^{FL}$ or $\bar{\Phi}_\alpha^{FLR}$

**Multiple tests**

Testing $(H_{0,m})$ $P \in \mathcal{P}_{0,m}$ simultaneously

Multiple tests whose $\mathrm{FWER} \leq \alpha$: $\mathcal{R}^{Bonf}$, $\mathcal{R}^{Holm}$, or $\mathcal{R}^{minp}$

Under specific conditions,
$\bar{\Phi}_\alpha^{Bonf} = \mathbb{1}_{\{\mathcal{R}^{Bonf}\neq\emptyset\}} = \mathbb{1}_{\{\mathcal{R}^{Holm}\neq\emptyset\}}$ and $\bar{\Phi}_\alpha^{FL} = \mathbb{1}_{\{\mathcal{R}^{minp}\neq\emptyset\}}$

✏ Fromont, Lerasle, Reynaud-Bouret, *Ann. Stat.* (2015)

# Multiple tests

## Multiple tests designed for particular concrete challenges

**Example:** Detecting synchronization periods between neural spike trains

➟ Multiple test based on permutation independence tests for point processes Case $[UD2]$

✏ Albert, Bouret, Fromont, Reynaud-Bouret, *Ann. Stat.* (2015)

✏ Albert, Bouret, Fromont, Reynaud-Bouret, *Neural Comp.* (minor rev, 2015)

**Perspectives:** Aggregation, study from the minimax point of view?

**Introduction of a minimax theory for multiple tests**

✏ Fromont, Lerasle, Reynaud-Bouret, *Ann. Stat.* (2015)

Allows to prove that when they are based on strongly dependent $p$ values, $\mathcal{R}^{Bonf}$ can be clearly suboptimal, whereas $\mathcal{R}^{minp}$ is minimax adaptive...

# Conclusion

**Aggregated or multiple tests** based on a collection of single tests, defined from test statistics $T_m$ (or $p$-values $p_m$) and associated critical values obtained from Monte Carlo or resampling methods, that are justified from a nonasymptotic point of view

$\Rightarrow$ implementable and adapted to moderate sample sizes

[$KD$] (Known Distr.) $\mathcal{L}_{(H_0)}(T_m)$ is known (parameter free)

[$UD$] (Unknown Distr.)

- [UD1] $\mathcal{L}_{(H_0)}(T_m|Z)$ is known
- [$UD$2] $\exists T_m^*$, $\mathcal{L}(T_m^*|Z) = \mathcal{L}_{(H_0)}(T_m|Z)$
- [$UD$3] $\exists T_m^*$, $\mathcal{L}_{(H_0)}(T_m^*|Z) = \mathcal{L}_{(H_0)}(T_m|Z)$

# Conclusion

[*KD*] Goodness-of-fit tests in the density model

✏ Fromont, Laurent, *Ann. Stat.* (2006)

Detection of atmospheric nitrogen deposition in ecology

[*KD*] Periodic signal detection tests in a regression model

✏ Fromont, Lévy-Leduc, *ESAIM P&S* (2006)

Target detection in laser vibrometry

[*UD*1] Homogenity tests in the Poisson model

✏ Fromont, Laurent, Reynaud-Bouret, *Ann. IHP* (2011)

Detection of epidemics

[*UD*2] Two-sample tests in the Poisson model

✏ Fromont, Laurent, Reynaud-Bouret, *Ann. Stat.* (2013)

Differential analysis of replication origins peaks in genetics

Spatial representativeness of services in public statistics

[*UD*2] Independence tests for point processes

✏ Albert, Bouret, Fromont, Reynaud-Bouret, *Ann. Stat.* (2013)

✏ Albert, Bouret, Fromont, Reynaud-Bouret, *Neural Comp.* (minor rev, 2015)

Detection of dependence periods between spike trains in neuroscience

[*UD*3] Two-sample tests in density and regression models

✏ Fromont, Laurent, Reynaud-Bouret, Lerasle, *COLT* (2012), Fromont, Tuleau (2015)

Comparison of functional data (in progress)